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Strong convergence and stability of Picard iteration sequences for a general class of contractive-type mappings

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Abstract

Let $(E, \|\cdot\|)$ be a normed linear space, $T : E \to E$ be a mapping of E into itself satisfying the following contractive condition: $\|T^i x - T^i y\| \le a^i \|x - y\| + \varphi_i(\|x - Tx\|)$, for each $x, y \in E$, $0 \le a^i < 1$, where $\varphi_i : \mathbb{R}^+ \to \mathbb{R}^+$ is a sub-additive monotone increasing function with $\varphi_i(0) = 0$ and $\varphi_i(Lu) = L\varphi_i(u)$, $L \ge 0$, $u \in \mathbb{R}^+$. It is shown that the Picard iteration process converges strongly to the unique fixed point of T. Furthermore, several classes of nonlinear operators studied by various authors are shown to belong to this class of mappings. Our theorem improves several recent important results. In particular, it improves a recent result of Akewe *et al.* (Fixed Point Theory Appl 2014:45, 2014), and a host of other results.

MSC: accretive-type mappings; pseudocontractive mappings; Picard sequence; contractive-type mappings

Keywords: 47H04; 47H06; 47H15; 47H17; 47J25

1 Nonlinear operators of accretive-type and fixed points

Let K be a nonempty subset of a real normed space E. A mapping $T : K \to E$ is called *Lipschitz* if there exists $L \ge 0$ such that

$$\|Tx - Ty\| \le L\|x - y\| \quad \forall x, y \in K. \tag{1.1}$$

If $L \in [0, 1)$, the map T is called a *contraction map*, and if $L = 1$, T is called *nonexpansive*.

Let H be a real Hilbert space; a mapping $A : H \to H$ is called *monotone* if

$$\langle Ax - Ay, x - y\rangle \ge 0 \quad \forall x, y \in H. \tag{1.2}$$

Let E^* denote the topological dual space of E. A map $J : E \to 2^{E^*}$ defined by

$$Jx = \left\{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2 \ \forall x \in E\right\}$$

is called the *normalized duality map on* E. It is well known that if E^* is strictly convex then J is single-valued. In the sequel, single-valued normalized duality map will be denoted by j. In real Hilbert spaces, the normalized duality map is the identity map. A mapping A with domain $D(A)$ and range $R(A)$ in E is called *accretive* if, for all $x, y \in D(A)$, the following inequality is satisfied:

$$\|x - y\| \leq \|x - y + s(Ax - Ay)\| \quad \forall s > 0. \tag{1.3}$$

As a consequence of a result of Kato [1], it follows from inequality (1.3) that A is accretive if, for each $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq 0, \tag{1.4}$$

where $J : E \to 2^{E^*}$ is the normalized duality map on E. It follows, again from inequality (1.3), that A is accretive if and only if $(I + sA)$ is *expansive* and, consequently, its inverse $(I + sA)^{-1}$ exists and is nonexpansive as a mapping from $R(I + sA)$ into $D(A)$, where $R(I + sA)$ denotes the range of $(I + sA)$. The range of $(I + sA)$ does not need to be all of E. This leads to the following definition.

Definition 1.1 An operator A is said to be m-accretive if A is accretive and the range of $(I + sA)$ is all of E for some $s > 0$.

It can be shown that if $R(I + sA) = E$ for *some* $s > 0$, then it holds for *all* $s > 0$. The operator $-\Delta$, where Δ denotes the Laplacian, is an m-accretive operator. Let $f : E \to \mathbb{R}$ be a convex functional on a real normed space E. The *subdifferential of* f, denoted by ∂f, is a map $\partial f : E \to 2^{E^*}$ defined for each $x \in E$ by

$$(\partial f)(x) = \{x^* \in E^* : f(y) \geq f(x) + \langle y - x, x^*\rangle \ \forall y \in E\}. \tag{1.5}$$

If $E = H$, a real Hilbert space, it is easy to show that the *subdifferential of* f is a *maximal monotone operator*. Furthermore, it follows from (1.5) that if zero is in the subdifferential of f at some $u^* \in E$, then u^* is a minimizer of f. Thus, for a convex functional f on a real Hilbert space, solving the inclusion

$$0 \in \partial f(x)$$

amounts to finding a minimizer of f. More generally, we have the inclusion

$$0 \in Ax, \tag{1.6}$$

where A is a maximal monotone operator is of great interest in nonlinear operator theory.

The accretive operators were introduced independently in 1967 by Browder [2] and Kato [1]. Interest in such mappings stems mainly from their firm connection with the existence theory for nonlinear equations of evolution in Banach spaces of the form

$$\frac{du}{dt} + Au = 0, \qquad u(0) = u_0, \tag{1.7}$$

where A is an accretive map on an appropriate Banach space. At equilibrium, $\frac{du}{dt} = 0$ and solving the equation

$$Au = 0, \tag{1.8}$$

where A is an accretive operator amounts to solving for the equilibrium points of the evolution system (1.7).

Browder converted (1.8) to a fixed point problem. He introduced an operator T defined as follows: $T := I - A$, where A is accretive and called such a T, *pseudocontractive*. It is clear that fixed points of T correspond to zeros of A.

Pseudo-contractive maps are not necessarily continuous. The map $T : [0,1] \to \mathbb{R}$ defined by

$$Tx = \begin{cases} x - \frac{1}{2}, & \text{if } x \in [0, \frac{1}{2}), \\ x - 1, & \text{if } x \in (\frac{1}{2}, 1] \end{cases}$$

is pseudocontractive but is neither nonexpansive nor continuous.

Existence of solutions of system (1.7) has been established. Browder [2] proved that the system is solvable if A is locally Lipschitzian and accretive on E, and utilizing the existence result for system (1.7), he proved that if A is locally Lipschitz and accretive on E, then A is m-accretive.

Martin [3] proved that if A is *continuous* and accretive on E, then A is m-accretive. Browder [2] further proved that if $A : E \to E$ is Lipschitz and strongly accretive (*i.e.*, there exists $k \in \mathbb{R}$ such that for each $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that $\langle Ax - Ay, j(x-y)\rangle \geq k\|x-y\|^2$) then A is surjective. This result was subsequently generalized by Deimling [4] to the *continuous* strongly accretive operators (see, *e.g.*, Deimling [5, Theorem 13.1]). For details of accretive and monotone operators, the reader may consult Reich [6, 7].

2 Iterative methods for solutions of certain nonlinear equations

We begin with the well-known and celebrated contraction mapping principle.

Theorem 2.1 (Contraction mapping principle) *Let (X, ρ) be a complete metric space and $T : X \to X$ be a contraction map of X into itself. Then*

(a) *T has a unique fixed point, say x^* in X;*

(b) *the sequence $\{x_n\}_{n=0}^{\infty}$ in X defined by $x_0 \in X$,*

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, 3, \ldots \tag{2.1}$$

converges to x^*.

Theorem 2.1 is, perhaps, the most important fixed point theorem. The sequence of the recursion formula (2.1) is called the *Picard sequence*.

One important (see, *e.g.*, [8, p.57]) class of nonlinear mappings generalizing the class of contraction mappings is the class of nonexpansive mappings. Readers interested in nonexpansive mappings may consult, for example, Goebel and Reich [9], Reich [10].

If K is a nonempty *compact convex* subset of $\mathbb{R}^2$ and $T : K \to K$ is *a nonexpansive map*, even with a unique fixed point, the Picard sequence defined by (2.1) may fail to converge to the fixed point. It suffices to take $K = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ and let $T : K \to K$ be a rotation of K about the origin of coordinates through a fixed angle θ, $0 < \theta < \frac{\pi}{2}$ (say). It is easy to check that T is nonexpansive, zero is the unique fixed point of T and that the Picard sequence (2.1) with $x_0 = (1, 0)$ fails to converge to zero.

Following research efforts by Mann [11], Krasnoselskii [12], Schaefer [13], Ishikawa [14], Edelstein [15–17], Reinermann [18], Edelstein and O'Brian [17], Chidume [19], and a host of other authors, the following recursion formula was developed and found to be effective for approximating fixed points of *nonexpansive mappings*.

Let K be a nonempty convex subset of a normed space E and $T: K \to K$ be a nonexpansive map. Let the sequence $\{x_n\}_{n=0}^{\infty}$ in K be defined by

$$x_{n+1} = (1 - c_n)x_n + c_n Tx_n, \quad x_0 \in K, n \in \mathbb{N}, \tag{2.2}$$

where $\{c_n\}$ is a sequence in $(0,1)$ satisfying the following conditions: (i) $\sum_{n=0}^{\infty} c_n = \infty$, (ii) $\lim_{n\to\infty} c_n = 0$. *If the sequence* $\{x_n\}_{n=0}^{\infty}$ *is bounded*, Ishikawa [14] proved that the sequence is an *approximate fixed point sequence* in the sense that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{2.3}$$

Edelstein and O'Brian [17] considered the recursion formula

$$x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, \quad x_0 \in K, n \in \mathbb{N}, \lambda \in (0,1), \tag{2.4}$$

where T maps K into K and proved that *if* K *is bounded*, then the convergence in (2.3) is uniform.

Chidume [19] considered the recursion formula (2.2), introduced the concept of admissible sequence, and proved that *if* K *is bounded*, then the convergence in (2.3) is uniform *for the sequence defined by* (2.2).

Remark 1 We note here that the recursion formula (2.2) which is certainly cumbersome when compared with Picard iteration was developed for the class of nonexpansive maps because the simpler Picard sequence will not always converge for nonexpansive maps. Furthermore, the recursion formula (2.2) can only yield the result that the sequence defined by (2.2) satisfies (2.3). In general, it does not yield convergence of the sequence to a fixed point of T. To obtain convergence to a fixed point of T, *some type of compactness condition* must be imposed either on K or on the map T (*e.g.*, T may be required to be *demicompact at zero*, or $(I - T)$ may be required to *map closed bounded subsets of* E *into closed subsets of* E, *etc.*; see, *e.g.*, Chidume [8]). The recursion formula (2.2) is now generally referred to as *Mann formula* in the light of Mann [11].

An important class of mappings generalizing the class of nonexpansive mappings is the class of *Lipschitz pseudocontractive maps*. It is not difficult to check that every nonexpansive map is a Lipschitz pseudocontraction. We have already given an example of a pseudocontractive map which is not even continuous. All attempts to use the Mann formula, which has been successfully employed for nonexpansive mappings, to approximate a fixed point of a Lipschitz pseudocontractive map even on a compact convex domain in a real Hilbert space, proved abortive. In 1974, Ishikawa [20] proved the following theorem.

Theorem IS *Let* K *be a nonempty compact convex subset of a real Hilbert space* H *and* $T: K \to K$ *be a Lipschitz pseudocontractive map. Let the sequence* $\{x_n\}_{n=0}^{\infty}$ *be defined by*

$x_0 \in K$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n, \tag{2.5}$$

$$y_n = (1-\beta_n)x_n + \beta_n Tx_n, \quad n \geq 1, \tag{2.6}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real sequences satisfying the following conditions*: (i) $0 \leq \alpha_n \leq \beta_n < 1$ $\forall n \geq 1$; (ii) $\sum \alpha_n\beta_n = \infty$; (iii) $\lim_{n\to\infty} \beta_n = 0$. *Then* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to a fixed point of* T.

Remark 2 It is clear that the recursion formulas (2.5) and (2.6) of the Ishikawa scheme are more cumbersome than the Mann formula (2.2). However, since it was not known whether or not the simpler Mann sequence would always converge to fixed points of Lipschitz pseudocontractive maps, the cumbersome Ishikawa scheme was applied for this class of maps. The question of whether or not the simpler Mann sequence had actually failed for this class of maps remained open for many years. This was resolved in 2001 by Chidume and Mutangadura [21] who produced an example of a Lipschitz pseudocontractive map defined on a compact convex subset of $\mathbb{R}^2$ with a unique fixed point for which no Mann sequence converges.

Remark 3 (a) We first observe that if we set $\beta_n = 0$ $\forall n$ in the recursion formula (2.6) then condition (i) in Theorem IS shows that $\alpha_n = 0$ $\forall n$ and so (2.5) and (2.6) reduce to $x_{n+1} = x_n$ $\forall n$, so that $\{x_n\}_{n=0}^{\infty}$ converges to x_0, the initial approximation which may not be a fixed point of T.

(b) Because the Ishikawa formulas were used successfully in approximating a fixed point of T in Theorem IS, several authors started studying a modification of it in which condition (i) is replaced by the condition: (i)* $0 \leq \alpha_n, \beta_n < 1$, and condition (ii) is modified accordingly. In this modification, α_n and β_n are independent and it is permissible to set $\beta_n = 0$ for all n. They still called such a modified formula an Ishikawa formula. *This is wrong*. To see this, it suffices to set $\beta_n = 0$ $\forall n$ and see that the sequence obtained from the modified scheme will not converge to a fixed point of T in Theorem IS. In particular, if $\beta_n = 0$ $\forall n$, *the modified formula generally reduces to the Mann formula* and then the example of Chidume and Mutangadura [21] shows that the modified formula will not converge to a fixed point of T in the setting of Theorem IS (see, *e.g.*, [8] for more comments on the Ishikawa iteration formula).

(c) The order of convergence of the Picard sequence is that of a geometric progression, that of the Mann sequence is of the form $O(\frac{1}{n})$, while that of the Ishikawa sequence is of the form $O(\frac{1}{\sqrt{n}})$. Furthermore, whenever Picard sequence converges, it is preferred to the Mann sequence which itself is preferred to the Ishikawa formula whenever it converges, because the preferred recursion formula is simpler (consequently requiring less computation and therefore reducing cost of computation).

Three other iteration methods have been introduced and have successfully been employed to approximate fixed points of Lipschitz pseudocontractive mappings even in *Banach spaces more general than Hilbert spaces*.

Let K be a nonempty closed convex and bounded subset of a Hilbert space H. Suppose that $T : K \to K$ is a pseudocontractive and Lipschitzian map with constant $L \geq 0$. For arbitrary $z_0, w \in K$, Schu [22] defined the following *two-step* iteration process:

$$z_{n+1} = (1 - \mu_{n+1})w + \mu_{n+1} y_n, \tag{2.7}$$

$$y_n = (1 - \alpha_n)w + \alpha_n T z_n, \tag{2.8}$$

where the real sequences $\{\mu_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ are in $(0,1)$ and satisfy appropriate conditions and are such that $(\{\mu_n\}, \{\alpha_n\})$ has *property* A (see Schu [22] for a definition). Schu proved that $\{z_n\}_{n=1}^{\infty}$ converges strongly to the unique fixed point of T nearest to w.

This result was extended by Chidume [23] to real Banach spaces possessing weakly sequential continuous duality maps (*e.g.*, l_p spaces, $1 < p < \infty$).

A second iteration scheme for approximating fixed points of Lipschitz pseudocontractive mappings was implicitly introduced by Bruck [24] who actually applied the scheme, still in Hilbert spaces, to approximate a solution of the inclusion $0 \in Ax$ where A is an *m-monotone operator*.

Let H be a Hilbert space, $A : H \to H$ be an *m-monotone operator* with $0 \in R(A)$, the range of A. For arbitrary $z \in H$, Bruck considered the sequence $\{x_n\}$ in H defined by $x_0 \in H$,

$$x_{n+1} = x_n - \lambda_n \big(Ax_n + \theta_n (x_n - x_1)\big), \tag{2.9}$$

and proved that if $\{x_n\}$ and $\{Ax_n\}$ are bounded, then $\{x_n\}$ converges strongly to some x^*, solution of $0 \in Au$, provided λ_n and θ_n are *acceptably paired sequences* (*e.g.*, see [24] for a definition).

An example of acceptably paired sequences given in [24] is $\lambda_n = n^{-1}$, $\theta_n = (\log(\log n))^{-1}$, $n(i) = i^i$.

The ideas of sequences with *property* A and sequences that are *acceptably paired* are due to Halpern [25]. Reich [26] also studied the recursion formula (2.9) for Lipschitz accretive operators on real uniformly convex Banach spaces with *a duality mapping that is weakly sequentially continuous at zero*.

Motivated by the papers of Abbas *et al.* [27], Chidume [28], Reich [26, 29, 30], Shahzad and Al-Dubiban [31], Chidume and Zegeye [32] studied the following perturbation of the Mann recurrence relation to approximate fixed points of Lipschitz pseudocontractive mappings in real Banach spaces much more general than Hilbert spaces. Let E be a real normed space, K be a nonempty convex subset of E, $T : K \to K$ be a Lipschitz pseudocontractive map. For arbitrary $x_1 \in K$, let the sequence $\{x_n\}$ be defined iteratively by

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T x_n - \lambda_n \theta_n (x_n - x_1),$$

where λ_n and θ_n are real sequences in $(0,1)$ satisfying appropriate conditions. They proved the following theorem.

Theorem CZ ([32]) *Let K be a nonempty closed convex subset of a real Banach space E. Let $T : K \to K$ be a Lipschitz pseudocontractive map with constant $L > 0$ and $F(T) := \{x \in K : Tx = x\} \neq \emptyset$. Let a sequence $\{x_n\}$ be generated from arbitrary $x_1 \in K$ by*

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T x_n - \lambda_n \theta_n (x_n - x_1), \tag{2.10}$$

for all positive integers n, where λ_n *and* θ_n *are real sequences in* $(0,1)$ *satisfying appropriate conditions.*

Then $\|x_n - Tx_n\| \to 0$ *as* $n \to \infty$.

Remark 4 Real sequences applicable in Theorem CZ are

$$\lambda_n = \frac{1}{(n+1)^a}, \qquad \theta_n = \frac{1}{(n+1)^b}, \quad \text{where } 0 < b < a \text{ and } a + b < 1.$$

Remark 5 We have shown in this section that a cumbersome recurrence relation is desirable and introduced *only when a simpler recurrence formula is not available for the class of mappings under consideration*. It is obvious that whenever a k-step method works for any class of maps, it is trivial to construct an n-step method that will work for the same class of maps, $n > k$, $n \in \mathbb{N}$. Such an n-step method will, in general, require more computation time and therefore will be less efficient than the k-step method. In general, the rate of convergence of such an n-step method is at best the same as that of the k-step method. Consequently, such n-step methods serve no useful purpose and are therefore not desirable.

3 A strong convergence theorem

Akewe *et al.* [33] stated the following theorem.

Theorem AOO ([33, Theorem 2.2, p.7]) *Let* $(E, \|\cdot\|)$ *be a normed linear space,* $T : E \to E$ *be a self-map of* E *satisfying the following contractive condition*:

$$\|T^i x - T^i y\| \le a^i \|x - y\| + \sum_{j=0}^{i} \binom{i}{j} a^{i-j} \varphi(\|x - Tx\|), \tag{3.1}$$

for each $x, y \in E$, $0 \le a^i < 1$, *where* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a sub-additive monotone increasing function with* $\varphi(0) = 0$ *and* $\varphi(Lu) = L\varphi(u)$, $L \ge 0$, $u \in \mathbb{R}^+$. *For* $x_0 \in E$, *let* $\{x_n\}_{n=0}^{\infty}$ *be the Kirk-multistep iterative scheme defined by*

$$\begin{aligned}
x_{n+1} &= \alpha_{n,0} x_n + \sum_{i=1}^{k_i} \alpha_{n,i} T^i y_n^i, \quad \sum_{i=1}^{k_i} \alpha_{n,i} = 1, \\
y_n^i &= \beta_{n,0} x_n + \sum_{i=1}^{k_{j+1}} \beta_{n,i}^j T^i y_n^{i+1}, \quad \sum_{i=0}^{k_{j+1}} \beta_{n,i}^j = 1, j = 1, 2, \ldots, q-2, \\
y_n^{q-1} &= \sum_{i=0}^{k_q} \beta_{n,i}^{q-1} T^i x_n, \quad \sum_{i=0}^{k_q} \beta_{n,i}^{q-1} = 1, q \ge 2, n \ge 0,
\end{aligned} \tag{3.2}$$

where $k_1 \ge k_2 \ge k_3 \ge \cdots \ge k_q$, *for each* $j, \alpha_{n,i} \ge 0$, $\alpha_{n,0} \ne 0$, $\beta_{n,j}^j \ge 0$, $\beta_{n,0}^j \ge 0$.

Then

(i) T *defined by* (3.1) *has a unique fixed point* p;

(ii) *the Kirk-multistep iterative scheme* (*defined by* (3.2)) *converges strongly to the fixed point* p *of* T.

Remark 6 The authors of Theorem AOO *did not prove* (i) as claimed. The *existence* of a fixed point of T was not proved. What the authors showed is that *if T has a fixed point*, then the fixed point is unique.

Remark 7 Kirk [34] introduced the following *one-step* iterative method for approximating a fixed point of a nonexpansive map T: $x_0 \in E$:

$$x_{n+1} = \sum_{i=0}^{k} \alpha_i T^i x_n, \quad n \geq 0, \sum_{i=0}^{k} \alpha_i = 1. \tag{3.3}$$

He never introduced the *multistep* method defined in (3.2).

Remark 8 In Theorem AOO, the summation $\sum_{j=0}^{i} \binom{i}{j} a^{i-j}$ is simply $(1+a)^i$. Therefore, the contractive condition (3.1) reduces to the following:

$$\|T^i x - T^i y\| \leq a^i \|x - y\| + (1+a)^i \varphi(\|x - Tx\|), \tag{3.4}$$

so that, setting $\varphi_i = (1+a)^i \varphi(\|x - Tx\|)$, we have $\varphi_i : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi_i(0) = 0$ and $\varphi_i(Lu) = L\varphi_i(u)$, $L \geq 0$, $u \in \mathbb{R}^+$ and thus obtain the following contractive-type condition:

$$\|T^i x - T^i y\| \leq a^i \|x - y\| + \varphi_i(\|x - Tx\|), \tag{3.5}$$

which is basically a compact form of (3.1).

Now, assuming the existence of a fixed point for a mapping T satisfying contractive condition (3.1) or (3.5), we prove that a Picard sequence for T^i converges strongly to the unique fixed point of T.

Theorem 3.1 *Let $(E, \|\cdot\|)$ be a real normed space and $T : E \to E$ be a map satisfying the contractive condition* (3.1) *or* (3.5), *with constant $a^i = a$. Assume that T has a fixed point $p \in E$. For arbitrary $x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined by*

$$x_{n+1} = T^i x_n, \quad n \geq 1, n \in \mathbb{N}. \tag{3.6}$$

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to p.

Proof Since $Tp = p$, put $x = p$ and $T^i = T$ in the contractive condition (3.1) or (3.5), to obtain

$$\|T^i x - p\| \leq a^i \|x - p\|, \tag{3.7}$$

for all $x \in E$, where $a^i \in [0,1)$. Using formula (3.6) and inequality (3.7), we obtain

$$\|x_{n+1} - p\| \leq (a^i)^n \|x_1 - p\| \to 0 \quad \text{as } n \to \infty.$$

Hence, $\{x_n\}_{n=0}^{\infty}$ converges strongly to p. This completes the proof. □

Another general class of mappings generalizing the contraction mappings was introduced by Hardy and Rogers [35] as follows: Let (M,d) be a complete metric space and $T : M \to M$ satisfy the following contractive condition: $\forall x, y \in M$,

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx),$$

where $a_i \geq 0$ $\forall i = 1, 2, 3, 4, 5$, $\sum_{i=1}^{5} a_i < 1$. Hardy and Rogers proved that T has a unique fixed point. Several authors proved fixed point theorems for mappings satisfying special cases of the contractive condition of Hardy and Rogers. We observe that if p denotes the unique fixed point of T in the theorem of Hardy and Rogers, then the following inequality holds:

$$d(Tx, p) \leq ad(x, p) \quad \forall x \in M,$$

where $a := \frac{a_1 + a_2 + a_3}{1 - a_2 - a_5} \in (0, 1)$.

Remark 9 If an operator satisfies inequality (3.7) where p is a fixed point of T, then p is necessarily unique. For assume that there exists $q \neq p$ such that $Tq = q$. Then $\|p - q\| = \|Tp - Tq\| \leq a^i \|p - q\|$ so that $(1 - a^i)\|p - q\| \leq 0$, which yields $p = q$.

Remark 10 The T-stability of the Picard iterative scheme, whenever it converges, is well known (see, *e.g.*, Ostrowski [36], Berinde [37, 38], Bruck [24], Rhoades [39], Harder and Hicks [40], Shahzad and Zegeye [41]).

Remark 11 In the light of Remark 5, our theorem is a significant improvement on the results of Akewe *et al.* [33] in the sense that the Picard sequence defined by (3.6) is much simpler than the multistep methods (3.2) considered in [33]. Furthermore, the Picard sequence converges as fast as a geometric progression whereas convergence with the multistep methods considered in [33] is either of order $O(\frac{1}{n})$ or of order $O(\frac{1}{\sqrt{n}})$.

Competing interests

The author declares that there are no competing interests.

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